

Bifurcation of asymptotically stable periodic solutions in nearly impact oscillators: DRAFT

Oleg Makarenkov

Department of Mathematics, Imperial College London, UK

Ferdinand Verhulst

Mathematical Institute, University of Utrecht, Netherlands

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1 Introduction

Since the observation by Glover-Lazer-McKenna [3] that a simple harmonic oscillator with a piecewise linear stiffness (jumping nonlinearity) contributes to the explanation of

the failure of the Takoma bridge, the studying of periodic oscillations in such models got a lot of attention of mathematicians; see the recent survey [8] and the papers [12, 2]. Also, new engineering studies of impact oscillators open up a large potential for challenging extensions of these results. In fact, Ivanov [4] argued that harmonic oscillators with a jumping nonlinearity with one part of the force field nearly infinite is a better model for describing the bouncing ball, rather than its limit version for an impact oscillator. In our modeling the resulting system of differential equations is singularly perturbed, but as we discuss below, the classical singular perturbation theory does not apply. In this paper we develop an averaging-like approach which solves the problem in a weakly nonlinear case. For a discussion of the use of averaging method for regular impacting systems we refer to [1]. To be explicit, our approach concerns the existence and stability of periodic oscillations of the following system

$$\begin{aligned} \ddot{x} + x &= \varepsilon f(t, x, \dot{x}, \varepsilon), & x > 0, \\ \ddot{x} + \frac{1}{\varepsilon^2(\omega_\varepsilon)^2}x &= g(t, x, \dot{x}, \varepsilon), & x < 0, \end{aligned} \quad (1.1)$$

where $f, g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1], \mathbb{R})$, $\varepsilon > 0$ is a small parameter, $\omega_\varepsilon \rightarrow \omega_0 \in \mathbb{R}$ as $\varepsilon \rightarrow 0$. System 1.1 can be considered as a smoothed version of a system with impacts. We will study resonance oscillations and assume, therefore, that

$$f(t + \pi, u, v, \varepsilon) \equiv f(t, u, v, \varepsilon), g(t + \pi, u, v, \varepsilon) \equiv g(t, u, v, \varepsilon).$$

System (1.1) represents a natural singular perturbation description of impact phenomena which is different from the usual approaches. Our main result (Theorem 1) states that the emergence of asymptotically stable π -periodic solutions in (1.1) from π -periodic cycles of non-smooth limiting the system

$$\begin{aligned} \ddot{x} + x &= 0, & x > 0, \\ \dot{x}(t - 0) &= -\dot{x}(t + 0), & x(t) = 0, \end{aligned} \quad (1.2)$$

can be studied by a special form of the averaging method combined with a suitable scaling of time when solutions pass the half plane $x < 0$. This involves the use of the implicit function theorem for a non-smooth problem in the limit as $\varepsilon \rightarrow 0$; this is possible by introducing a suitable Poincaré map. The result is a change of stability of a fixed point when its eigenvalues enter the unit disc from outside through the imaginary axis.

2 Main result

We prove the following theorem.

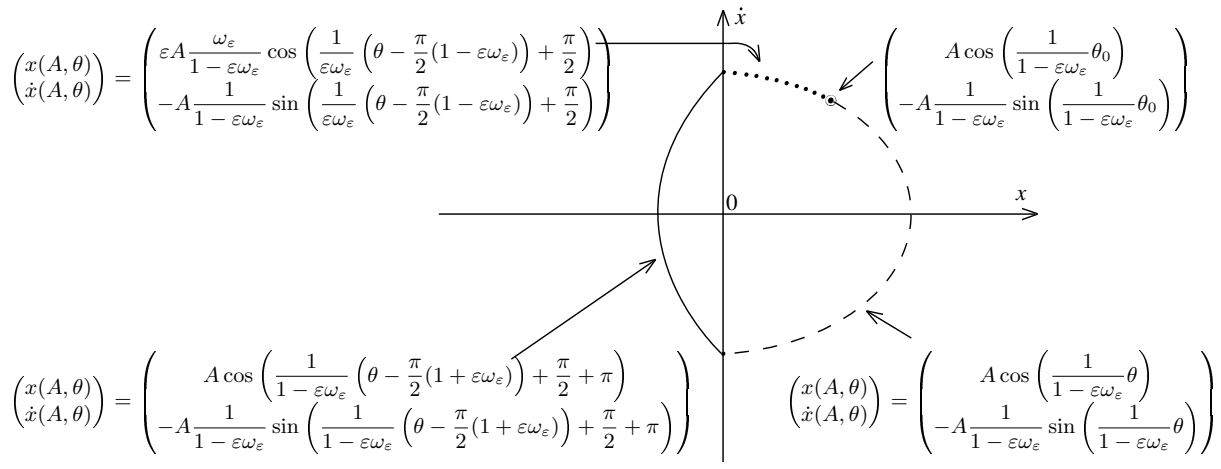


Fig. 1: Illustration of the change of variables (2.5)-(2.7).

Theorem 2.1 Let $f, g \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ be π -periodic with respect to time. Define

$$\begin{aligned} \overline{P}(A, \theta) = & - \int_0^{\frac{\pi}{2}-\theta} \left(\frac{1}{A} \cos(\tau + \theta) \right) (f(\tau, A \cos(\tau + \theta), -A \sin(\tau + \theta), 0) - 2\omega_0 A \cos(\tau + \theta)) d\tau - \\ & - \int_{\frac{\pi}{2}-\theta}^{\pi} \left(\frac{1}{A} \cos(\tau + \theta + \pi) \right) (f(\tau, A \cos(\tau + \theta + \pi), -A \sin(\tau + \theta + \pi), 0) - \\ & - 2\omega_0 A \cos(\tau + \theta + \pi)) d\tau - \omega_0 \int_0^{\pi} \left(\sin\left(s + \frac{\pi}{2}\right) \right) g\left(\frac{\pi}{2} - \theta, 0, -A \sin\left(s + \frac{\pi}{2}\right), 0\right) ds. \end{aligned}$$

If $\overline{P}(A_0, \theta_0) = 0$ for some $(A_0, \theta_0) \in \mathbb{R} \times (0, \pi)$ and the real parts of eigenvalues of $\overline{P}'(A_0, \theta_0)$ are negative, then, for any $\varepsilon > 0$ sufficiently small, equation (1.1) has a unique π -periodic solution satisfying

$$x_\varepsilon(t) \rightarrow x_0(t) \quad \text{as } \varepsilon \rightarrow 0 \quad \text{pointwise on } [0, \pi] \setminus \left\{ \frac{\pi}{2} - \theta_0 \right\}$$

where x_0 is the unique π -periodic solution of the equation (1.2) with initial condition

$$(x(0), \dot{x}(0)) = (A_0 \cos \theta_0, -A_0 \sin \theta_0).$$

Moreover, the solution x_ε is asymptotically stable.

Proof. Rewrite system (1.1) as follows (see Fig. 2)

$$\ddot{x} + \frac{1}{(1 - \varepsilon \omega_\varepsilon)^2} x = \varepsilon f(t, x, \dot{x}, \varepsilon) - 2\varepsilon \frac{\omega_\varepsilon}{(1 - \varepsilon \omega_\varepsilon)^2} x - \varepsilon^2 \frac{(\omega_\varepsilon)^2}{(1 - \varepsilon \omega_\varepsilon)^2} x, \quad x > 0, \quad (2.3)$$

$$\ddot{x} + \frac{1}{(\varepsilon)^2 (\omega_\varepsilon)^2} x = g(t, x, \dot{x}, \varepsilon), \quad x < 0, \quad (2.4)$$

so, that any solution of the reduced system ($\varepsilon = 0$)

$$\begin{aligned} \ddot{x} + \frac{1}{(1 - \varepsilon \omega_\varepsilon)^2} x &= 0, \quad x > 0, \\ \ddot{x} + \frac{1}{(\varepsilon)^2 (\omega_\varepsilon)^2} x &= 0, \quad x < 0 \end{aligned}$$

is π -periodic. Let us introduce variables (A, θ) as follows

$$\begin{cases} x &= A \cos\left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \theta\right), \\ \dot{x} &= -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin\left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \theta\right), \quad \theta \in \left[0, \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon)\right], \end{cases} \quad (2.5)$$

$$\begin{cases} x &= \varepsilon A \frac{\omega_\varepsilon}{1 - \varepsilon \omega_\varepsilon} \cos \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right), \\ \dot{x} &= -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right), \quad \theta \in \left[\frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon), \frac{\pi}{2} (1 + \varepsilon \omega_\varepsilon) \right], \end{cases} \quad (2.6)$$

$$\begin{cases} x &= A \cos \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 + \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right), \\ \dot{x} &= -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 + \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right), \quad \theta \in \left[\frac{\pi}{2} (1 + \varepsilon \omega_\varepsilon), \pi \right], \end{cases} \quad (2.7)$$

which transforms equations (2.3)-(2.4) to the following system

$$\begin{pmatrix} \dot{A} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon G_1(t, A, \theta, \varepsilon), \quad \text{if } \theta \in \left[0, \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right], \quad (2.8)$$

$$\begin{pmatrix} \dot{A} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon G_2(t, A, \theta, \varepsilon), \quad \text{if } \theta \in \left[\frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon), \frac{\pi}{2} (1 + \varepsilon \omega_\varepsilon) \right], \quad (2.9)$$

$$\begin{pmatrix} \dot{A} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon G_3(t, A, \theta, \varepsilon), \quad \text{if } \theta \in \left[\frac{\pi}{2} (1 + \varepsilon \omega_\varepsilon), \pi \right], \quad (2.10)$$

where

$$G_1(t, A, \theta, \varepsilon) = \begin{pmatrix} -(1 - \varepsilon \omega_\varepsilon) \sin \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \theta \right) \left(f \left(t, A \cos \left(\frac{\theta}{1 - \varepsilon \omega_\varepsilon} \right), \right. \right. \\ \left. \left. -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{\theta}{1 + \varepsilon \omega_\varepsilon} \right), \varepsilon \right) - \right. \\ \left. \left. -A \frac{\omega_\varepsilon}{(1 - \varepsilon \omega_\varepsilon)^2} (2 + \varepsilon \omega_\varepsilon) \cos \left(\frac{\theta}{1 - \varepsilon \omega_\varepsilon} \right) \right) \right) \\ - \frac{(1 - \varepsilon \omega_\varepsilon)^2}{A} \cos \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \theta \right) \left(f \left(t, A \cos \left(\frac{\theta}{1 - \varepsilon \omega_\varepsilon} \right), \right. \right. \\ \left. \left. -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{\theta}{1 + \varepsilon \omega_\varepsilon} \right), \varepsilon \right) - \right. \\ \left. \left. -A \frac{\omega_\varepsilon}{(1 - \varepsilon \omega_\varepsilon)^2} (2 + \varepsilon \omega_\varepsilon) \cos \left(\frac{\theta}{1 - \varepsilon \omega_\varepsilon} \right) \right) \right) \end{pmatrix}$$

$$G_2(t, A, \theta, \varepsilon) = \begin{pmatrix} -\frac{1}{\varepsilon} (1 - \varepsilon \omega_\varepsilon) \sin \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right) \cdot \\ \cdot g \left(t, \varepsilon A \frac{\omega_\varepsilon}{1 - \varepsilon \omega_\varepsilon} \cos \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right), \right. \\ \left. -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right), \varepsilon \right) \\ - \frac{1}{A} (1 - \varepsilon \omega_\varepsilon) \omega_\varepsilon \cos \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right) \cdot \\ \cdot g \left(t, \varepsilon A \frac{\omega_\varepsilon}{1 - \varepsilon \omega_\varepsilon} \cos \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right), \right. \\ \left. -A \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{1}{\varepsilon \omega_\varepsilon} \left(\theta - \frac{\pi}{2} (1 - \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} \right), \varepsilon \right) \end{pmatrix}$$

$$G_3(t, A, \theta, \varepsilon) = \begin{pmatrix} -(1 - \varepsilon\omega_\varepsilon) \sin \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right) \cdot \\ \cdot \left(f \left(t, A \cos \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) \right) + \frac{\pi}{2} + \pi \right), \right. \\ \left. -A \frac{1}{1 - \varepsilon\omega_\varepsilon} \sin \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right), \varepsilon \right) - \\ -A \frac{\omega_\varepsilon}{(1 - \varepsilon\omega_\varepsilon)^2} (2 + \varepsilon\omega_\varepsilon) \cos \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right) \\ - \frac{(1 - \varepsilon\omega_\varepsilon)^2}{A} \cos \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right) \cdot \\ \cdot \left(f \left(t, A \cos \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) \right) + \frac{\pi}{2} + \pi \right), \right. \\ \left. -A \frac{1}{1 - \varepsilon\omega_\varepsilon} \sin \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right), \varepsilon \right) - \\ \left. -A \frac{\omega_\varepsilon}{(1 - \varepsilon\omega_\varepsilon)^2} (2 + \varepsilon\omega_\varepsilon) \cos \left(\frac{1}{1 - \varepsilon\omega_\varepsilon} \left(\theta - \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right) \right). \end{pmatrix}$$

If $t \mapsto (A(t), \theta(t) - t)$ is an asymptotically stable π -periodic solution of (2.8)-(2.10) then (x, \dot{x}) defined by (2.5)-(2.7) is an asymptotically stable π -periodic solution of (2.3)-(2.4). To prove the existence of asymptotically stable π -periodic solutions of equations (2.8)-(2.10) we show that each solution $(\bar{A}(\cdot, A, \theta, \varepsilon), \bar{\theta}(\cdot, A, \theta, \varepsilon))$ of (2.8)-(2.10) with initial condition $(\bar{A}(0, A, \theta, \varepsilon), \bar{\theta}(0, A, \theta, \varepsilon)) = (A, \theta)$ is defined on $[0, \pi]$ whenever (A, θ) belongs to a small neighborhood of (A_0, θ_0) , and that the map

$$P_\varepsilon(A, \theta) = (\bar{A}(\pi, A, \theta, \varepsilon), \bar{\theta}(\pi, A, \theta, \varepsilon) - \pi) \quad (2.11)$$

contracts in this neighborhood.

Step 1. First we show that solution $t \mapsto (\bar{A}(\cdot, A, \theta, \varepsilon), \bar{\theta}(\cdot, A, \theta, \varepsilon))$ of (2.8)-(2.10) on $[0, \pi]$ can be consequently sewed by solutions of systems (2.8), (2.9) and (2.10).

Denote by $t \mapsto (\bar{A}_i(\cdot, t_0, A, \theta, \varepsilon), \bar{\theta}_i(\cdot, t_0, A, \theta, \varepsilon))$, $i = 1, 2, 3$, the solutions of (2.8), (2.9), (2.10) respectively with initial condition $(\bar{A}_i(t_0, t_0, A, \theta, \varepsilon), \bar{\theta}_i(t_0, t_0, A, \theta, \varepsilon)) = (A, \theta)$. Put

$$F_1(T, A, \theta, \varepsilon) = \frac{1}{1 - \varepsilon\omega_\varepsilon} \bar{\theta}_1(T, 0, A, \theta, \varepsilon) - \frac{\pi}{2}.$$

Since $F_1\left(\frac{\pi}{2} - \theta_0, A_0, \theta_0, 0\right) = \bar{\theta}_1\left(\frac{\pi}{2} - \theta_0, 0, A_0, \theta_0, 0\right) - \frac{\pi}{2} = 0$ and

$$(F_1)'_T\left(\frac{\pi}{2} - \theta_0, A_0, \theta_0, 0\right) = (\bar{\theta})'_{(1)}\left(\frac{\pi}{2} - \theta_0, 0, A_0, \theta_0, 0\right) = 1$$

then by the implicit function theorem [5, Ch. X, § 2, Theorems 1 and 2] there exists $T_1 \in C^1(\mathbb{R} \times \mathbb{R} \times [0, 1], \mathbb{R})$ such that

$$F_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon) = 0, \quad |A - A_0| < \delta, \quad |\theta - \theta_0| < \delta, \quad \varepsilon \in [0, \delta),$$

where $\delta > 0$ sufficiently small. Or, equivalently,

$$\frac{1}{1 - \varepsilon\omega_\varepsilon} \bar{\theta}_1(T_1(A, \theta, \varepsilon), 0, A, \theta, \varepsilon) = \frac{\pi}{2}, \quad |A - A_0| < \delta, \quad |\theta - \theta_0| < \delta, \quad \varepsilon \in [0, \delta).$$

Therefore, the solution of system (2.8) with initial condition (A, θ) at $t = 0$ approaches the threshold of switching to (2.9) at time $T_1(A, \theta, \varepsilon)$.

Now we show that the solution

$$\begin{pmatrix} \bar{A}_2\left(\cdot, T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon\right), \\ \bar{\theta}_2\left(\cdot, T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon\right) \end{pmatrix}$$

stays till some time $T_2(A, \theta, \varepsilon)$ in

$$[0, \infty) \times \left[\frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon) \right]$$

and that $T_2(A, \theta, \varepsilon)$ is given by

$$T_2(A, \theta, \varepsilon) = T_1(A, \theta, \varepsilon) + \varepsilon \tilde{T}_2(A, \theta, \varepsilon),$$

where $\tilde{T}_2 \in C^1(\mathbb{R} \times \mathbb{R} \times [0, 1], \mathbb{R})$. To do this consider

$$\begin{aligned} F_2(T, A, \theta, \varepsilon) &= \frac{1}{\varepsilon \omega_\varepsilon} \left(\bar{\theta}_2(T_1(A, \theta, \varepsilon) + \varepsilon T, T_1(A, \theta, \varepsilon), \right. \\ &\quad \left. \bar{A}_1(T_1(A, \theta, \varepsilon), 0, A, \theta, \varepsilon), \bar{\theta}_1(T_1(A, \theta, \varepsilon), 0, A, \theta, \varepsilon), \varepsilon) - \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon) \right) - \pi. \\ F_2(T, A, \theta, \varepsilon) &= \begin{cases} \frac{1}{\varepsilon \omega_\varepsilon} \left(\bar{\theta}_2(T_1(A, \theta, \varepsilon) + \varepsilon T, T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \right. \\ \quad \left. \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon), \varepsilon) - \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon) \right) - \pi, & \text{if } \varepsilon > 0, \\ \frac{1}{\omega_0} T - \pi, & \text{if } \varepsilon = 0. \end{cases} \end{aligned}$$

Let us convince ourself that the function F_2 verifies the assumptions of the implicit function theorem [5, Ch. X, § 2, Theorems 1 and 2] at the point $(T, A, \theta, \varepsilon) = (\omega_0 \pi, A_0, \theta_0, \varepsilon)$. Since

$$\frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon) = \bar{\theta}_2 \left(T_1(A, \theta, \varepsilon), T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon), \varepsilon) \right)$$

then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_2(T, A, \theta, \varepsilon) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \omega_\varepsilon} (\bar{\theta}_2)'_{(1)}(T_1(A, \theta, \varepsilon) + \lambda(A, \theta, \varepsilon) \varepsilon T, T_1(A, \theta, \varepsilon), \\ &\quad \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon), \varepsilon) \varepsilon T - \pi = \frac{1}{\omega_0} T - \pi, \end{aligned}$$

that is F_2 is continuous at $\varepsilon = 0$. Here $\lambda(A, \theta, \varepsilon) \in [0, 1]$. Furthermore, we have

$$(F_2)'_T \left(\frac{\pi}{2} - \theta_0 + \pi, A_0, \theta_0, 0 \right) = \frac{1}{\omega_0} \neq 0.$$

Therefore, the implicit function theorem [5, Ch. X, § 2, Theorems 1 and 2] allows as to conclude that there exists $\tilde{T}_2 \in C^1(\mathbb{R} \times \mathbb{R} \times [0, 1], \mathbb{R})$ such that

$$\begin{aligned} \frac{1}{\varepsilon \omega_\varepsilon} \left(\bar{\theta}_2(T_1(A, \theta, \varepsilon) + \varepsilon \tilde{T}_2(A, \theta, \varepsilon), T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \right. \\ \left. \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon), \varepsilon) - \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon) \right) = \pi, \quad |A - A_0| < \delta, |\theta - \theta_0| < \delta, \varepsilon \in [0, \delta). \end{aligned}$$

Since $\theta_0 \in (0, \pi)$, then $\delta > 0$ can be diminished in such a way that

$$\begin{aligned} &\bar{\theta}_3(\pi, T_1(A, \theta, \varepsilon) + \varepsilon \tilde{T}_2(A, \theta, \varepsilon), \bar{A}_2(T_1(A, \theta, \varepsilon) + \varepsilon \tilde{T}_2(A, \theta, \varepsilon), \\ &\quad T_1(A, \theta, \varepsilon), \bar{A}_1(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon \omega_\varepsilon), \varepsilon), \frac{\pi}{2}(1 + \varepsilon \omega_\varepsilon), \varepsilon) < \\ &< \frac{\pi}{2} - \theta + \pi + \pi, \quad \text{for } \varepsilon > 0 \text{ sufficiently small} \end{aligned}$$

and any $|A - A_0| < \delta, |\theta - \theta_0| < \delta$. Therefore, the solution of (2.10) with initial conditions under consideration does not meet the line of discontinuity during time $(T_1(A, \theta, \varepsilon) + \varepsilon \tilde{T}_2(A, \theta, \varepsilon), \pi]$.

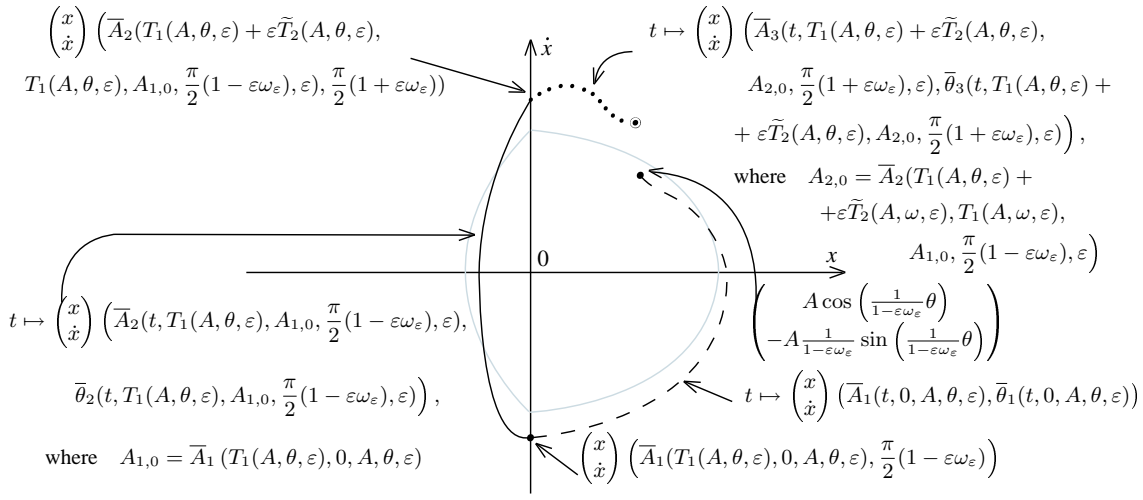


Fig. 2: Illustration of sewing of the solution of system (2.8)-(2.8) by solutions of each its single component (2.8), (2.9) and (2.10).

Summarizing, we can define the solution $t \mapsto (\bar{A}(t, A, \theta, \varepsilon), \bar{\theta}(t, A, \theta, \varepsilon))$ of system (2.8)-(2.10) as follows (see Fig. 2)

$$(\bar{A}(t, A, \theta, \varepsilon), \bar{\theta}(t, A, \theta, \varepsilon)) = \begin{cases} \begin{pmatrix} \bar{A}(t, A, \theta, \varepsilon) \\ \bar{\theta}(t, A, \theta, \varepsilon) \end{pmatrix}, & \text{if } t \in [0, T_1(A, \theta, \varepsilon)], \\ \begin{pmatrix} \bar{A}_2(t, T_1(A, \theta, \varepsilon), \bar{A}(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon) \\ \bar{\theta}_2(t, T_1(A, \theta, \varepsilon), \bar{A}(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon), \varepsilon) \end{pmatrix}, & \text{if } t \in [T_1(A, \theta, \varepsilon), T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon)], \\ \begin{pmatrix} \bar{A}_3(t, T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon), \bar{A}(T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon), \varepsilon) \\ \bar{\theta}_3(t, T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon), \bar{A}(T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon), A, \theta, \varepsilon), \frac{\pi}{2}(1 + \varepsilon\omega_\varepsilon), \varepsilon) \end{pmatrix}, & \text{if } t \in [T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon), \pi]. \end{cases}$$

Step 2. At this step we show that fixed points of the Poincaré map (2.11) can be studied by means of the function \bar{P} introduced in the formulation of the theorem. To this end we decompose P_ε as

$$P_\varepsilon(a, \theta) = \begin{pmatrix} A \\ \theta \end{pmatrix} + \varepsilon(\bar{P}_{\varepsilon,1}(A, \theta) + \bar{P}_{\varepsilon,2}(A, \theta) + \bar{P}_{\varepsilon,3}(A, \theta)),$$

where

$$\begin{aligned} \bar{P}_{\varepsilon,1}(A, \theta) &= \int_0^{T_1(A, \theta, \varepsilon)} G_1(\tau, \bar{A}(\tau, A, \theta, \varepsilon), \bar{\theta}(\tau, A, \theta, \varepsilon), \varepsilon) d\tau, \\ \bar{P}_{\varepsilon,2}(A, \theta) &= \int_{T_1(A, \theta, \varepsilon)}^{T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon)} G_2(\tau, \bar{A}(\tau, A, \theta, \varepsilon), \bar{\theta}(\tau, A, \theta, \varepsilon), \varepsilon) d\tau, \\ \bar{P}_{\varepsilon,3}(A, \theta) &= \int_{T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon)}^\pi G_3(\tau, \bar{A}(\tau, A, \theta, \varepsilon), \bar{\theta}(\tau, A, \theta, \varepsilon), \varepsilon) d\tau. \end{aligned}$$

Since \sin , \cos and g are bounded on any bounded set then from system (2.8)-(2.10) we have that

$$\begin{pmatrix} \bar{A}(t, A, \theta, \varepsilon) \\ \bar{\theta}(t, A, \theta, \varepsilon) \end{pmatrix} \rightarrow \begin{pmatrix} A \\ t + \theta \end{pmatrix} \quad \text{as } \varepsilon \rightarrow 0$$

uniformly with respect to $t \in [0, \pi]$, $|A - A_0| < \delta$, $|\theta - \theta_0| < \delta$. This gives us immediately that

$$\begin{aligned}\overline{P}_{\varepsilon,1}(A, \theta) &\rightarrow \int_{T_1(A, \theta, 0)}^{T_1(A, \theta, 0)} G_1(\tau, A, \tau + \theta, 0) d\tau, \\ \overline{P}_{\varepsilon,3}(A, \theta) &\rightarrow \int_{T_1(A, \theta, 0)}^{\pi} G_3(\tau, A, \tau + \theta, 0) d\tau, \quad \text{as } \varepsilon \rightarrow 0,\end{aligned}\tag{2.12}$$

uniformly with respect to $|A - A_0| < \delta$, $|\theta - \theta_0| < \delta$. Since we proved that T_1 and \tilde{T}_2 are continuously differentiable, then (2.12) implies that

$$\begin{aligned}(\overline{P}_{\varepsilon,1})'(A, \theta) &\rightarrow (P_{0,1})'(A, \theta), \\ (\overline{P}_{\varepsilon,3})'(A, \theta) &\rightarrow (P_{0,3})'(A, \theta), \quad \text{as } \varepsilon \rightarrow 0,\end{aligned}$$

uniformly with respect to $|A - A_0| < \delta$, $|\theta - \theta_0| < \delta$.

Let us now study the behavior of $\overline{P}_{\varepsilon,2}$ and $(\overline{P}_{\varepsilon,2})'$ as $\varepsilon \rightarrow 0$. We have

$$\begin{aligned}\overline{P}_{\varepsilon,2}(A, \theta) &= -(1 - \varepsilon\omega_\varepsilon) \int_{T_1(A, \theta, \varepsilon)}^{T_1(A, \theta, \varepsilon) + \varepsilon\tilde{T}_2(A, \theta, \varepsilon)} \left(\frac{1}{\varepsilon} \sin \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(\tau, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right) \right. \\ &\quad \left. \frac{1}{\overline{A}(\tau, A, \theta, \varepsilon)} (1 - \varepsilon\omega_\varepsilon)\omega_\varepsilon \cos \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(\tau, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right) \right. \\ &\quad \left. \cdot g \left(\tau, \varepsilon\overline{A}(\tau, A, \theta, \varepsilon) \frac{\omega_\varepsilon}{1 - \varepsilon\omega_\varepsilon} \cos \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(\tau, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right), \right. \right. \\ &\quad \left. \left. - \overline{A}(\tau, A, \theta, \varepsilon) \frac{1}{1 - \varepsilon\omega_\varepsilon} \sin \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(\tau, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right), \varepsilon \right) d\tau.\end{aligned}$$

Scaling the time in the integral as $\tau = T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s$ we get

$$\begin{aligned}\overline{P}_{\varepsilon,2}(A, \theta) &= -\omega_\varepsilon(1 - \varepsilon\omega_\varepsilon) \int_0^{T_2(A, \theta, \varepsilon)\omega_\varepsilon} \left(\sin \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right) \right. \\ &\quad \left. \frac{1}{\varepsilon\overline{A}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon)} (1 - \varepsilon\omega_\varepsilon)\omega_\varepsilon \cdot \cos \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right) \right. \\ &\quad \left. \cdot g \left(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, \varepsilon\overline{A}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) \frac{\omega_\varepsilon}{1 - \varepsilon\omega_\varepsilon} \cos \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right) \right. \right. \\ &\quad \left. \left. + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right), -\overline{A}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) \frac{1}{1 - \varepsilon\omega_\varepsilon} \cdot \sin \left(\frac{1}{\varepsilon\omega_\varepsilon} \left(\bar{\theta}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) + \frac{\pi}{2} \right), \varepsilon \right) ds.\end{aligned}$$

Put

$$K_\varepsilon(A, \theta) = \frac{1}{\varepsilon} \left(\bar{\theta}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \bar{\theta}(T_1(A, \theta, \varepsilon), A, \theta, \varepsilon) \right).$$

Since

$$\begin{aligned}\frac{1}{\varepsilon} \left(\bar{\theta}(T_1(A, \theta, \varepsilon) + \varepsilon\omega_\varepsilon s, A, \theta, \varepsilon) - \frac{\pi}{2}(1 - \varepsilon\omega_\varepsilon) \right) &= \\ = K_\varepsilon(A, \theta) &\rightarrow (\bar{\theta})'_{(1)}(T_1(A, \theta, 0), A, \theta, 0)\omega_0 s = \omega_0 s, \quad \text{as } \varepsilon \rightarrow 0,\end{aligned}$$

then

$$\overline{P}_{\varepsilon,2}(A, \theta) \rightarrow -\omega_0 \int_0^\pi \left(\sin \left(s + \frac{\pi}{2} \right) \right) g \left(\frac{\pi}{2} - \theta, 0, A \sin \left(s + \frac{\pi}{2} \right), 0 \right) ds, \quad \text{as } \varepsilon \rightarrow 0,$$

uniformly with respect to $|A - A_0| < \delta, |\theta - \theta_0| < \delta$. Since $(K_\varepsilon)'(A, \theta)$ converges as $\varepsilon \rightarrow 0$ uniformly in $|A - A_0| < \delta$ and $|\theta - \theta_0| < \delta$ then $(K_\varepsilon)'(A, \theta) \rightarrow (K_0)'(A, \theta)$ as $\varepsilon \rightarrow 0$. Therefore,

$$(\overline{P}_{\varepsilon,2})'(A, \theta) \rightarrow (\overline{P}_{0,2})'(A, \theta) \quad \text{as } \varepsilon \rightarrow 0$$

uniformly with respect to $|A - A_0| < \delta, |\theta - \theta_0| < \delta$.

Summarizing, we proved, that

$$\begin{aligned} \frac{1}{\varepsilon} \left(P_\varepsilon(A, \theta) - \begin{pmatrix} A \\ \theta \end{pmatrix} \right) &= \overline{P}(A, \theta) + \\ &+ (\overline{P}_{\varepsilon,1} - \overline{P}_{0,1})(A, \theta) + \overline{P}_{\varepsilon,2}(A, \theta) - \overline{P}_{0,2}(A, \theta) + \overline{P}_{\varepsilon,3}(A, \theta) - \overline{P}_{0,3}(A, \theta). \end{aligned}$$

Therefore, for any $\varepsilon > 0$ sufficiently small, the function

$$(A, \theta) \mapsto P_\varepsilon(A, \theta) - \begin{pmatrix} A \\ \theta \end{pmatrix}$$

has a unique zero $(A_\varepsilon, \theta_\varepsilon)$ such that $(A_\varepsilon, \theta_\varepsilon) \rightarrow (A_0, \theta_0)$ as $\varepsilon \rightarrow 0$ and the real parts of eigenvalues of $(P_\varepsilon)'(A_\varepsilon, \theta_\varepsilon) - I$ are negative. This is equivalent to say that the eigenvalues of the matrix $((P_\varepsilon)'(A_\varepsilon, \theta_\varepsilon))^2$ belong to the interval $[0, 1)$. This implies (see [6, Lemma 9.2]) that $t \mapsto (\overline{A}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon), \overline{\theta}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) - t)$ is an asymptotically stable π -periodic solution of (2.8)-(2.10). To see the latter one will probably wish to make the change of variables $\Xi(t) = \theta(t) - t$ in (2.8)-(2.10). Since the change of variables (2.5)-(2.7) is π -periodic, than given by these formulas corresponding solution $(x_\varepsilon, \dot{x}_\varepsilon)$ of (2.3)-(2.4) is also π -periodic and asymptotically stable. Uniqueness of x_ε follows from uniqueness of $(A_\varepsilon, \theta_\varepsilon)$ and the fact that the change of variables (2.5)-(2.7) is one-to-one.

To finish the proof it remains to observe that for any $t \in [0, \frac{\pi}{2} - \theta_0]$ we have

$$\begin{pmatrix} \overline{A}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) \cos \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \overline{\theta}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) \right) \\ -A(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) \frac{1}{1 - \varepsilon \omega_\varepsilon} \sin \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \overline{\theta}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) \right) \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \cos(t + \theta_0) \\ -A_0 \sin(t + \theta_0) \end{pmatrix} \quad \text{as } \varepsilon \rightarrow 0$$

and that for any $t \in (\frac{\pi}{2} - \theta_0, \pi]$ we have

$$\begin{aligned} &\begin{pmatrix} \overline{A}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) \cos \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \left(\overline{\theta}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) - \frac{\pi}{2}(1 + \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right) \\ -A(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) \sin \left(\frac{1}{1 - \varepsilon \omega_\varepsilon} \left(\overline{\theta}(t, A_\varepsilon, \theta_\varepsilon, \varepsilon) - \frac{\pi}{2}(1 + \varepsilon \omega_\varepsilon) \right) + \frac{\pi}{2} + \pi \right) \end{pmatrix} \rightarrow \\ &\rightarrow \begin{pmatrix} A_0 \cos(t + \theta_0 + \pi) \\ -A_0 \sin(t + \theta_0 + \pi) \end{pmatrix} \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

that is x_ε converges to the solution of (1.2) with the initial condition $x_\varepsilon(0) = (A_0 \cos \theta_0, -A_0 \sin \theta_0)$ as $\varepsilon \rightarrow 0$ pointwise on $[0, \pi] \setminus \{t_0\}$. The proof is complete. \square

3 An application

In this section we apply the result of section 2 to an impact oscillator shown in Fig. 3.

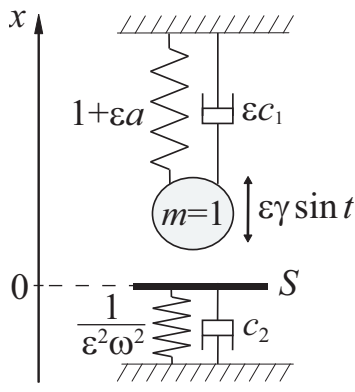


Fig. 3: Model of a preloaded ball (body of mass $m = 1$) bouncing against a nearly elastic surface S which is represented by a spring of stiffness $\frac{1}{\varepsilon^2\omega^2}$ with $\varepsilon > 0$ small. The rest coordinate of the mass is assumed to coincide with the origin of the x -axis on the one hand and with the surface S at rest on the other hand. We also suppose that the viscous friction coefficient equals εc_1 outside the contact with S and that it takes the value $c_2 + \varepsilon c_1$ during the contact.

A body of mass $m = 1$ is bouncing against a nearly elastic surface S (large stiffness $1/\varepsilon^2\omega^2$). Assuming in addition that the body is subjected to Rayleigh excitation, viscous friction and forcing, the equation of motions can be written as follows

$$\begin{aligned} \ddot{x} + x &= -\varepsilon a x - \varepsilon c_1 \dot{x} + \varepsilon \mu_1 \dot{x}(1 - \dot{x}^2) + \varepsilon \gamma \sin t, & \text{if } x \geq 0, \\ \ddot{x} + \frac{1}{\varepsilon^2\omega^2}x &= -(c_2 + \varepsilon c_1)\dot{x} + (\mu_2 + \varepsilon \mu_1)\dot{x}(1 - \dot{x}^2) + \varepsilon \gamma \sin t, & \text{if } x < 0. \end{aligned} \quad (3.13)$$

System (3.13) is of the form (1.1) with limiting system. There is no obvious reason why the Rayleigh excitation should be $O(1)$ during impact, but as this does not complicate the analysis, we admit this possibility. The averaging function \overline{P} takes now the longer form

$$\begin{aligned} \overline{P}(A, \theta) &= - \int_0^{\frac{\pi}{2}-\theta} \left(\frac{\sin(\tau + \theta)}{\frac{1}{A} \cos(\tau + \theta)} \right) (-aA \cos(\tau + \theta) + c_1 A \sin(\tau + \theta) + \\ &\quad - \mu_1 A \sin(\tau + \theta)(1 - A^2 \sin^2(\tau + \theta)) + \gamma \sin \tau - 2\omega A \cos(\tau + \theta)) d\tau - \\ &\quad - \int_{\frac{\pi}{2}-\theta}^{\pi} \left(\frac{\sin(\tau + \theta + \pi)}{\frac{1}{A} \cos(\tau + \theta + \pi)} \right) (-aA \cos(\tau + \theta + \pi) + c_1 A \sin(\tau + \theta + \pi) + \\ &\quad - \mu_1 A \sin(\tau + \theta + \pi)(1 - A^2 \sin^2(\tau + \theta + \pi)) + \gamma \sin \tau - 2\omega A \cos(\tau + \theta + \pi)) d\tau - \\ &\quad - \omega \int_0^{\pi} \left(\frac{\sin(\tau + \frac{\pi}{2})}{0} \right) \left(c_2 A \sin\left(\tau + \frac{\pi}{2}\right) - \mu_2 A \sin\left(\tau + \frac{\pi}{2}\right) \left(1 - A^2 \sin^2\left(\tau + \frac{\pi}{2}\right)\right) \right) d\tau = \\ &= \left(\gamma \theta \cos \theta - \frac{\pi}{2} A (c_1 + c_2 \omega) + \frac{\pi}{2} (\mu_1 + \mu_2 \omega) A \left(1 - \frac{3}{4} A^2\right) \right. \\ &\quad \left. - \frac{1}{A} \gamma \cos \theta - \frac{1}{A} \gamma \theta \sin \theta + \frac{\pi}{2} (a + 2\omega) \right). \end{aligned}$$

To formulate our result we need some preliminary notations. First, we introduce the function $M : \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} M(\theta) &= -\theta \cos \theta + (c_1 + c_2 \omega) \frac{\cos \theta + \theta \sin \theta}{a + 2\omega} + \\ &\quad - (\mu_1 + \mu_2 \omega) \left(\frac{\cos \theta + \theta \sin \theta}{a + 2\omega} - \frac{3\gamma^2}{\pi^2} \left(\frac{\cos \theta + \theta \sin \theta}{a + 2\omega} \right)^3 \right). \end{aligned}$$

Proposition 3.1 *Let $M(\theta_0) = 0$ for some $\theta_0 \in (0, \frac{\pi}{2})$ and A_0 is defined as*

$$A_0 = \frac{\gamma \cos \theta_0 + \gamma \theta_0 \sin \theta_0}{\frac{\pi}{2} (a + 2\omega)}. \quad (3.14)$$

If

$$M'(\theta_0) > 0 \quad (3.15)$$

and

$$-(c_1 + c_2 \omega) + (\mu_1 + \mu_2 \omega)(1 - 3A_0^2) < 0$$

then, for any $\varepsilon > 0$ sufficiently small, equation (3.13) has exactly one π -periodic solution x_ε such that

$$(x_\varepsilon(0), \dot{x}_\varepsilon(0)) \rightarrow (A_0 \cos \theta_0, -A_0 \sin \theta_0) \text{ as } \varepsilon \rightarrow 0.$$

The solution x_ε is asymptotically stable.

The proof of proposition (3.1) relies on the following lemma.

Lemma 3.2 Consider 2×2 real matrix D . If $Sp D < 0$ and $\det \|D\| > 0$ then the eigenvalues of D have negative real parts.

The statement of the lemma follows from the direct computation of the eigenvalues of D according to the standard formula for roots of quadratic equations.

Proof of proposition 3.1. Direct computation shows that (A_0, θ_0) is a zero of \overline{P} . To prove the proposition it remains to show that

$$1) Sp \overline{P}'(A_0, \theta_0) < 0.$$

$$2) \det \|\overline{P}'(A_0, \theta_0)\| > 0.$$

But these two relations follow from the formulas

$$1) Sp \overline{P}'(A_0, \theta_0) = -\pi(c_1 + c_2\omega) + \pi(\mu_1 + \mu_2\omega)(1 - 3A_0^2).$$

$$2) \frac{2A_0}{\pi\gamma(a + 2\omega)} \det \|\overline{P}'(A_0, \theta_0)\| = M'(\theta_0),$$

which are straightforward. □

Our next proposition 3.3 shows that proposition 3.1 is not vacuous, namely we give sufficient conditions ensuring that (3.15) is satisfied. Before proceeding to the formulation of proposition 3.3 we need to introduce some notations and properties. First we observe, that the

$$\frac{M(\theta)}{\theta \cos \theta} = -1 + K(\theta),$$

where

$$K(\theta) = \left(\frac{c_1 + c_2\omega}{a + 2\omega} - \frac{\mu_1 + \mu_2\omega}{a + 2\omega} \left(1 - \frac{3\gamma^2}{\pi^2} \left(\frac{\cos \theta + \theta \sin \theta}{a + 2\omega} \right)^2 \right) \right) \left(\frac{1}{\theta} + tg \theta \right).$$

Observe, that there exists $\theta_* \in (0, \frac{\pi}{2})$ such that

$$K'(\theta) \neq 0 \text{ for all } \theta \in \left(\theta_*, \frac{\pi}{2} \right). \quad (3.16)$$

In fact, if $K'(\theta_n) = 0$ for some sequence $\theta_n \uparrow \frac{\pi}{2}$ as $n \rightarrow \infty$, then

$$M'(\theta_n) = (\cos \theta_n - \theta_n \sin \theta_n)(-1 + K(\theta_n)) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

that contradicts the boundedness of the derivative of M .

Proposition 3.3 Let $\theta_* \in (0, \frac{\pi}{2})$ be such a number that (3.16) holds true. Assume that

$$K(\theta_*) < 1$$

and denote by $\theta_0 \in (\theta_*, \frac{\pi}{2})$ the unique point satisfying

$$-1 + K(\theta_0) = 0. \quad (3.17)$$

Then $M(\theta_0) = 0$ and $M'(\theta_0) < 0$. Consequently, for any $\varepsilon > 0$ sufficiently small, equation (3.13) has exactly one π -periodic solution x_ε such that

$$(x_\varepsilon(0), \dot{x}_\varepsilon(0)) \rightarrow (A_0 \cos \theta_0, -A_0 \sin \theta_0) \text{ as } \varepsilon \rightarrow 0,$$

where

$$A_0 = \frac{\gamma \cos \theta_0 + \gamma \theta_0 \sin \theta_0}{\frac{\pi}{2}(a + 2\omega)}.$$

The solution x_ε is asymptotically stable.

Note that $\theta_0 \in (\theta_*, \frac{\pi}{2})$ satisfying (3.17) always exists and is unique since $-1 + K(\theta_*) < 0$, $K'(\theta) \neq 0$, for $\theta \in (\theta_*, \frac{\pi}{2})$, and

$$\lim_{\uparrow \frac{\pi}{2}} K(\theta) = +\infty. \quad (3.18)$$

Proof of proposition 3.3. First, observe that $M(\theta_0) = \theta_0 \cos \theta_0 (-1 + K(\theta_0)) = 0$. Second, we have

$$M'(\theta) = (\cos \theta - \theta \sin \theta)L(\theta) + \theta \cos \theta K'(\theta)$$

and so $M'(\theta_0) = K'(\theta_0)$. But properties (3.16) and (3.18) imply that $K'(\theta_0) > 0$ and so the proof is complete. \square

Let us finally formulate our result in the simpler setting when the Rayleigh excitation is switched off, that is $\mu_1 = \mu_2 = 0$. We have that

$$K(\theta) = \frac{c_1 + c_2\omega}{a + 2\omega} \left(\frac{1}{\theta} + tg(\theta) \right),$$

in particular, there exists an unique $\theta_* \in (0, \frac{\pi}{2})$ such that $\text{sign } K'(\theta_*) = \text{sign } (\theta - \cos \theta) = 0$.

Proposition 3.4 Let $\theta_* \in (0, \frac{\pi}{2})$ be the unique point such that $\theta_* - \cos \theta_* = 0$. Assume that

$$\frac{c_1 + c_2\omega}{a + 2\omega} \left(\frac{1}{\theta_*} + tg \theta_* \right) < 1.$$

Denote by $\theta_0 \in (\theta_*, \frac{\pi}{2})$ the unique point such that

$$-1 + \frac{c_1 + c_2\omega}{a + 2\omega} \left(\frac{1}{\theta_0} + tg \theta_0 \right) = 0.$$

Then the conclusion of proposition 3.3 holds true.

4 Discussion

- It is remarkable that the analysis of system (1.1) can be handled by the introduction of a Poincaré map and the use of the implicit function theorem although the limiting system for $\varepsilon \rightarrow 0$ is non-smooth.
- At the same time we have formulated an unusual type of singular perturbation problem. Putting $\varepsilon = 0$, we have non-smooth impact, for $\varepsilon > 0$ we have fast motion in a neighborhood of the subset $x = 0$. For $x > 0$ slow motion takes place but this is not described by standard slow manifold theory, see [11]. Still, the dynamics for $x > 0$ can be considered as taking place in an explicitly formulated slow manifold. On the other hand, the solutions for $x < 0$ have as slow manifold the boundary $x = 0$. This does not satisfy the necessary hyperbolicity condition, but the solutions for $x > 0$ are forced to the manifold $x = 0$ and, after passing by a fast transition through the domain $x < 0$ they are forced again to leave $x = 0$. We note also that sliding along the slow manifold, as happens for instance in dry friction problems, is not possible. This simplifies the bifurcational behavior.
- Regarding the averaging result obtained in this paper, we draw attention to the papers [10]- [9] and further references there. In [10] the framework of differential inclusions is used, in [9] explicit estimates of the vector field and the solutions are given in the case of impulsive forces. Our approach economically avoids the estimate of general solution behavior as we aim at the more restricted result of obtaining periodic solutions.

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